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On the quantisation of Arnold's cat

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Abstract. We characterise the quantisation U_A of the classical map $A \in \mathrm{SL}(2, \mathbb{Z})$ using the Heisenberg group, construct the eigenstates for N = perfect square (where $\hbar = 2\pi/N$) and show that the Fourier components of the Wigner functions of a complete set of eigenstates go to zero for $N = p^2$, p prime, $p \to \infty$, A hyperbolic.

1. Introduction

In order to study the quantum mechanics of systems whose classical motion is chaotic, it is useful to examine the quantisation of simple classical maps. The maps we are considering here are the area-preserving linear maps of the two-dimensional torus onto itself ('Arnold's cat' (Arnold and Avez 1968)). They are described by 2×2 matrices A which have integer elements and determinant 1 (i.e. $A \in \mathrm{SL}\ (2, \mathbb{Z})$).

These maps were first quantised by Hannay and Berry (1980). Due to the compactness of the phase space the dimension of the Hilbert space of quantum states is finite. Its dimension N is related to Planck's constant by $\hbar = 2\pi/N$, where we are assuming the periods of the torus to equal 2π.

Other maps which have been quantised so far are the standard map on the torus (Izrailev 1986, 1987) and the Baker's transformation (Balazs and Voros 1987, 1989).

Important properties of quantum systems are the distribution of eigenvalues and the behaviour of the eigenfunctions. The energy level spacing distribution, for instance, of the two quantised maps just mentioned is in good agreement with the generic one (namely the GOE for these maps), in contrast to the cat maps (for a discussion of the eigenvalues of the quantised cat see Hannay and Berry (1980)).

If the classical dynamics of a system are completely chaotic, one might expect that in the semiclassical limit the eigenfunctions of the quantised system look very irregular. Berry (1977) conjectured for such a system that the smoothed Wigner function of each eigenstate converges to the classical microcanonical distribution for $\hbar \to 0$ and that the eigenfunctions behave like Gaussian random functions (see also Voros (1979)). In fact, it is known that the Wigner functions of almost all eigenstates converge to the classical distribution, if the classical motion is ergodic (Shnirelman 1974, Helffer *et al* 1987 and references therein). The question, however, as to whether this is true for each individual eigenstate is much more subtle. After numerical computations for the quantum stadium billiard by McDonald (1983) and Taylor and Brumer (1983) it became clear that the conjectured picture of the eigenfunctions has to be modified (Heller 1984). They found that some states look very regular even at high energies and that they are localised in some part of the configuration space (for instance, in

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the rectangular region of the stadium or in channels along closed classical orbits exhibiting 'scars'). Good approximations for some of the regular states were found by Shapiro *et al* (1984) and Bai *et al* (1985) using suitable Born-Oppenheimer approximations. 'Scars' were also found for a quartic oscillator (Eckhardt *et al* 1989) and for the Baker's transformation (Balazs and Voros 1989). In this paper we will show that no such localisation persists for $N \to \infty$ for the quantised cat (A hyperbolic) under the additional assumption $N = (\text{prime})^2$. More precisely, we prove that the Wigner functions converge weakly to equidistribution. Essentially the same result was also found by Eckhardt (1986) using less rigorous arguments.

A theory of the contribution of closed classical orbits to the eigenfunctions was developed by Bogumolny (1988) and extended to Wigner functions by Berry (1989). Their work is similar in spirit to the analysis of Gutzwiller (1971) of the Green function as a sum over classical paths. This theory was applied to cat maps by Keating (1989). The results in Bogumolny (1988) and Berry (1989) seem to imply that the contribution of scars to the Wigner functions tends to zero for $\hbar \to 0$. Note, however, that the size of the considered energy interval has to be sufficiently small in order to resolve individual eigenstates and that in this limit the convergence of the involved closed orbit series is questionable, as mentioned in Berry (1989).

In section 2 we will reformulate the quantisation of A in a more algebraic setting using the Heisenberg group.

In Hannay and Berry (1980) the quantum propagator U_A was constructed in terms of the classical action. We will characterise U_A by the transformation behaviour of the Heisenberg group under U_A, namely (2.7). In both approaches one uses the fact that semiclassical approximations are exact due to the linearity of A.

In section 3 we calculate all eigenvectors and eigenvalues of U_A for the case that N is a perfect square. One way to construct eigenvectors of U_A (for special A) was described by Eckhardt (1986) (see also Esposti and Knauf 1989). The key is to find states such that one can write the eigenfunctions as superpositions of these states in a simple way. It is interesting that our states are completely different from those used by Eckhardt and that they are delocalised in position as well as in momentum.

One important question is, of course, whether the assumption $N = (\text{prime})^2$ is only of a technical nature or whether number theoretical properties of N play a crucial role. We will discuss this briefly at the end of section 4.

2. The quantisation and the Heisenberg group

The Heisenberg group is defined by

$$t(x)t(y) = \exp(-\mathrm{i}\pi x \wedge y)t(x+y) \qquad x, y \in \mathbb{R}^2 \tag{2.1}$$

where

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \wedge \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = x_1 y_2 - x_2 y_1,$$

and the $t(x)$ are unitary operators on some Hilbert space.

We are interested in the subalgebra generated by

$$t_1 := t\left(\frac{1}{\sqrt{N}} e_1\right) \qquad t_2 := t\left(\frac{1}{\sqrt{N}} e_2\right) \tag{2.2}$$

where

$$e_1 = \begin{pmatrix}1\\0\end{pmatrix} \qquad e_2 = \begin{pmatrix}0\\1\end{pmatrix} \qquad N \in \mathbb{N}.$$

This is the largest subalgebra which leaves the considered Hilbert space invariant.

We get a representation of this algebra in terms of the position operator q and the momentum operator p by

$$t_1 = e^{iq} \qquad t_2 = e^{ip}. \tag{2.3}$$

This is consistent with (2.1) and (2.2) if we choose $\hbar = 2\pi/N$.

All finite dimensional unitary irreducible representations of this algebra are determined by two phases φ_1, φ_2 in the following way:

$$t_i^N = \exp(2\pi i \varphi_i) \cdot \mathbb{1} \qquad i = 1, 2 \tag{2.4}$$

and there exists an orthogonal basis $\{\psi_k\}_{k=0}^{N-1}$, such that

$$t_1 \psi_k = \exp\left(\frac{2\pi i}{N}(k + \varphi_1)\right)\psi_k \qquad t_2 \psi_k = \exp\left(\frac{2\pi i}{N}\varphi_2\right)\psi_{k-1} \tag{2.5}$$

where $\psi_{k+N} := \psi_k$.

If we make the identification (2.3), then we may represent ψ_k as a wavefunction by

$$\psi_k(q) = \sum_{m=-\infty}^{\infty} \delta\left(q - \frac{2\pi}{N}(k + \varphi_1) - 2\pi m\right)\exp(i\varphi_2 q). \tag{2.6}$$

Then ψ_k is periodic (up to the phases φ_1, φ_2) in q and in p with period 2π. (The ψ_k are of course not normalisable in $L^2(\mathbb{R})$ and the scalar product has to be 'renormalised' such that the ψ_k form an orthonormal basis.)

We will now see that U_A is completely determined up to a phase by the requirement

$$U_A^* \exp[i(kq + lp)]\, U_A = \exp\left[i(kl)A\begin{pmatrix}q\\p\end{pmatrix}\right] \qquad k, l \in \mathbb{Z}. \tag{2.7}$$

In other words, $\exp[i(kq + lp)]$ should transform as an operator under U_A in the same way as the corresponding phase space function under the classical map.

The reason that (2.7) should hold exactly is that we are quantising a linear map. (The same relation holds for the harmonic oscillator.) Because of (2.2) and (2.3) we may write (2.7) as

$$U_A^* t(x)\, U_A = t(A^{\mathrm{T}} x) \qquad x \in \frac{1}{\sqrt{N}}\mathbb{Z}^2. \tag{2.8}$$

For (2.8) to be valid it is sufficient that

$$U_A^* t_i U_A = t\left(\frac{1}{\sqrt{N}} A^{\mathrm{T}} e_i\right) \qquad i = 1, 2.$$

The general case then follows for $x = (1/\sqrt{N})\,(x_1 e_1 + x_2 e_2)$, $x_1, x_2 \in \mathbb{Z}$ from

$$\begin{aligned} U_A^* t(x)\, U_A &= \exp\left(\frac{i\pi}{N} x_1 x_2\right)(U_A^* t_1 U_A)^{x_1}(U_A^* t_2 U_A)^{x_2} \\ &= \exp\left(\frac{i\pi}{N} x_1 x_2\right) U_A^* t\left(\frac{x_1}{\sqrt{N}} A^{\mathrm{T}} e_1\right) t\left(\frac{x_2}{\sqrt{N}} A^{\mathrm{T}} e_2\right) U_A \\ &= U_A^* t(A^{\mathrm{T}} x)\, U_A \end{aligned}$$

since

$$A^{\mathrm{T}}e_1 \wedge A^{\mathrm{T}}e_2 = e_1 \wedge e_2 = 1.$$

Let now

$$\tilde{t}_i = t\left(\frac{1}{\sqrt{N}} A^{\mathrm{T}} e_i\right)$$

Because of det $A = 1$ we have

$$\tilde{t}_1 \tilde{t}_2 = \exp\left(-\frac{2\pi \mathrm{i}}{N}\right) \tilde{t}_2 \tilde{t}_1 \tag{2.9}$$

and the $\tilde{t}_i$ generate therefore the same algebra as the t_i. For (2.8) to be consistent it must hold that

$$\tilde{t}_i^N = \exp(2\pi \mathrm{i} \varphi_i) \cdot \mathbb{1} \qquad i = 1, 2. \tag{2.10}$$

On the other hand, if (2.10) is satisfied, then U_A exists and is unique up to a phase. To see this, we note that (2.9) and (2.10) imply that there exists an orthogonal basis $\{\tilde{\psi}_k\}_{k=0}^{N-1}$, such that (2.5) holds with t_i and ψ_k replaced by $\tilde{t}_i$, $\tilde{\psi}_k$, respectively.

Now (2.8) implies

$$t_1 U_A \tilde{\psi}_0 = U_A \tilde{t}_1 \tilde{\psi}_0 = \exp\left(\frac{2\pi \mathrm{i}}{N} \varphi_1\right) U_A \tilde{\psi}_0 \tag{2.11}$$

and therefore

$$U_A \tilde{\psi}_0 = \mathrm{e}^{\mathrm{i}\varphi} \psi_0 \tag{2.12}$$

for some phase φ. But then

$$U_A \tilde{\psi}_k = \exp\left(\frac{2\pi \mathrm{i}}{N} k\varphi_2\right) U_A \tilde{t}_2^{-k} \tilde{\psi}_0 = \exp\left(\frac{2\pi \mathrm{i}}{N} k\varphi_2\right) t_2^{-k} U_A \tilde{\psi}_0 = \mathrm{e}^{\mathrm{i}\varphi} \psi_k \tag{2.13}$$

and U_A is uniquely determined. On the other hand, if we define U_A by (2.13), then

$$U_A^* t_i U_A = \tilde{t}_i \qquad i = 1, 2 \tag{2.14}$$

and (2.8) holds.

It remains to investigate under what conditions for A (2.10) holds. But

$$\begin{aligned} \tilde{t}_1^N &= t(\sqrt{N} A^{\mathrm{T}} e_1) \\ &= \exp(\mathrm{i}\pi N A_{11} A_{12}) t_1^{N A_{11}} t_2^{N A_{12}} \\ &= \exp(\mathrm{i}\pi N A_{11} A_{12}) \exp\ 2\pi \mathrm{i}(A_{11}\varphi_1 + A_{12}\varphi_2) \cdot \mathbb{1}. \end{aligned} \tag{2.15}$$

and

$$\tilde{t}_2^N = \exp(\mathrm{i}\pi N A_{21} A_{22}) \exp 2\pi \mathrm{i}(A_{21}\varphi_1 + A_{22}\varphi_2) \cdot \mathbb{1}. \tag{2.16}$$

We therefore get the quantisation condition

$$(A - \mathbb{1})\begin{pmatrix}\varphi_1 \\ \varphi_2\end{pmatrix} = \tfrac{1}{2} N \begin{pmatrix} A_{11} \cdot A_{12} \\ A_{21} \cdot A_{22}\end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \qquad n_1, n_2 \in \mathbb{Z} \tag{2.17}$$

and we have proven the following theorem.

Theorem 2.1. Consider the N-dimensional unitary representation of the algebra generated by t_1 and t_2 which satisfies (2.4). Let $A \in \mathrm{SL}(2, \mathbb{Z})$ and assume that (2.17) holds for some integers n_1, n_2. Then there exists an up to a phase unique unitary map U_A such that (2.8) holds.

Let us make some remarks to the condition (2.17). This is a generalisation of the 'checkerboard' condition imposed by Hannay and Berry, namely that

$$A = \begin{pmatrix} \text{odd} & \text{even} \\ \text{even} & \text{odd} \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} \text{even} & \text{odd} \\ \text{odd} & \text{even} \end{pmatrix}. \tag{2.18}$$

In this case we may simply choose $\varphi_1 = \varphi_2 = 0$.

This is also true for N even. If N is odd, choose, for instance, $\varphi_1 = 0$, $\varphi_2 = \frac{1}{2}$, if

$$A = \begin{pmatrix} \text{odd} & \text{odd} \\ \text{even} & \text{odd} \end{pmatrix} \qquad \text{and} \qquad \varphi_1 = \varphi_2 = \tfrac{1}{2}$$

if

$$A = \begin{pmatrix} \text{even} & \text{odd} \\ \text{odd} & \text{odd} \end{pmatrix}.$$

This generalisation was also obtained by Esposti and Knauf (1989).

Note that different solutions of (2.17) lead to the same U_A. The only difference is, that $t_1, t_2, \tilde{t}_1, \tilde{t}_2$ have to be multiplied by some phase factors (t_i and $\tilde{t}_i$ by the same phase factor because of (2.17)).

For the sake of simplicity we will for the rest of this paper always assume that (2.18) holds and that $\varphi_1 = \varphi_2 = 0$.

To see that our U_A and the U_A chosen by Hannay and Berry coincide is an easy exercise.

The latter is defined by

$$\langle \psi_{k_2} | U_A | \psi_{k_1} \rangle = \left(\frac{\mathrm{i} A_{12}}{N} \right)^{1/2} \left\langle \exp\left(\frac{\mathrm{i}\pi}{N A_{12}} \{ A_{11}(k_1 + mN)^2 - 2(k_1 + mN)k_2 + A_{22}k_2^2 \} \right) \right\rangle_m \tag{2.19}$$

where $\langle \ldots \rangle_m$ denotes the average over all integers m. One has to show that

$$\langle \psi_{k_2} | t_1 U_A | \psi_{k_1} \rangle = \exp\left(\frac{2\pi \mathrm{i}}{N} k_2 \right) \langle \psi_{k_2} | U_A | \psi_{k_1} \rangle = \langle \psi_{k_2} | U_A \tilde{t}_1 | \psi_{k_1} \rangle$$

$$= \langle \psi_{k_2} | U_A | \psi_{k_1 - A_{12}} \rangle \exp\left(\frac{2\pi \mathrm{i}}{N} (k_1 - A_{12}) A_{11} \right) \exp\left(\mathrm{i} \frac{\pi}{N} A_{11} A_{12} \right)$$

and similarly that

$$\langle \psi_{k_2} | t_2 U_A | \psi_{k_1} \rangle = \langle \psi_{k_2} | U_A \tilde{t}_2 | \psi_{k_1} \rangle.$$

This is a straightforward computation.

The following properties of U_A, which are shown in Hannay and Berry (1980), are easily derived within our formalism. From (2.8) it follows immediately that $U_A U_B = U_{AB}$ (at least up to a phase). Furthermore one is interested in the smallest positive integer $n(N)$ such that

$$U_A^{n(N)} = \mathrm{e}^{\mathrm{i}\varphi} \cdot \mathbb{1} \tag{2.20}$$

for some phase φ. This is equivalent to

$$U_C^* t_i U_C = t_i \qquad (i=1,2) \qquad \text{where } C = A^{n(N)}.$$

But

$$U_C^* t_i U_C = \exp\left(\frac{i\pi}{N} C_{i1} C_{i2}\right) t_1^{C_{i1}} t_2^{C_{i2}}$$

and therefore it must hold $C \equiv \mathbb{1} \bmod N$ and (since we assume $\varphi_1 = \varphi_2 = 0$), B_{12} $(1+NB_{11})$ and B_{21} $(1+NB_{22})$ must be even, where $C = \mathbb{1} + NB$. The second condition follows from (2.18), if N is odd. If N is even then B_{21} and B_{12} must be even, and we get

$$\begin{aligned} A^{n(N)} &= \mathbb{1} + N \begin{pmatrix} \text{integer} & \text{integer} \\ \text{integer} & \text{integer} \end{pmatrix} \qquad \text{for } N \text{ odd} \\ A^{n(N)} &= \mathbb{1} + N \begin{pmatrix} \text{integer} & \text{even} \\ \text{even} & \text{integer} \end{pmatrix} \qquad \text{for } N \text{ even.} \end{aligned} \tag{2.21}$$

3. Eigenvectors and eigenvalues of U_A

First of all we construct an orthogonal basis which is well adapted to our problem. We assume that N is a square number, i.e. $N = p^2$, $p \in \mathbb{N}$. Then $t(e_1)$ and $t(e_2)$ are in the algebra generated by t_1 and t_2. Since they commute they can be diagonalised simultaneously. Since we assume $\varphi_1 = \varphi_2 = 0$ there exists a state ψ with

$$t(e_i)\psi = \psi \qquad i = 1, 2. \tag{3.1}$$

We can express ψ by ψ_k (see (2.5)) by

$$\psi = \frac{1}{\sqrt{p}} \sum_{j=0}^{p-1} \psi_{jp}. \tag{3.2}$$

Now define

$$\psi(x) := t(x)\psi \qquad x \in \frac{1}{p}\mathbb{Z}^2. \tag{3.3}$$

Then

$$t(e_i)\psi(x) = \exp(-2\pi i e_i \wedge x)\psi(x) \tag{3.4}$$

and therefore $\langle\psi(x)|\psi(y)\rangle = 0$ for $x - y \notin \mathbb{Z}^2$, since $\psi(x)$ and $\psi(y)$ are then eigenstates to different eigenvalues of $t(e_1)$ or $t(e_2)$. It follows that the $N = p^2$ states $\psi(x)$, $x = (1/p)$ $(x_1 e_1 + x_2 e_2)$, $0 \leqslant x_1, x_2 \leqslant p-1$, form an orthogonal basis.

Next we observe that

$$t(e_i) U_A \psi = U_A \psi \qquad i = 1, 2 \tag{3.5}$$

since

$$t(e_i) U_A \psi = U_A t(A^{\mathrm{T}} e_i)\psi = \exp(i\pi A_{i1} A_{i2}) U_A t(e_1)^{A_{i1}} t(e_2)^{A_{i2}} \psi = U_A \psi \tag{3.6}$$

because of (3.1) and (2.18).

But from (3.4) it follows that ψ is the only state satisfying (3.1). Therefore

$$U_A\psi = e^{i\gamma}\psi \tag{3.7}$$

for some phase γ. We choose $\gamma = 0$.

This implies now

$$U_A\psi(x) = (U_A t(x) U_A^*) U_A\psi = t(\tilde{A}x)\psi = \psi(\tilde{A}x) \tag{3.8}$$

where

$$\tilde{A} = (A^{\mathrm{T}})^{-1} \qquad x \in \frac{1}{p}\mathbb{Z}^2. \tag{3.9}$$

It turns now out that all eigenstates of U_A correspond to the closed orbits of $\tilde{A}$ modulo p on the lattice $\mathbb{Z}^2$. In fact, fix $x \in (1/p)\mathbb{Z}^2$ and let n be the smallest positive integer such that

$$x - \tilde{A}^n x \in \mathbb{Z}^2. \tag{3.10}$$

Then U_A acts like a shift operator times a phase on the subspace spanned by the orthogonal system $\{\psi(\tilde{A}^k x) \mid k = 0, 1, \ldots, n-1\}$. The phase is determined by the φ given by

$$\psi(\tilde{A}^n x) = \exp(2\pi i\varphi)\psi(x). \tag{3.11}$$

We obtain n eigenstates χ_k, $k = 0, \ldots, n-1$, by

$$\chi_k = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \exp\left(-\frac{2\pi i}{n}(k+\varphi)j\right)\psi(\tilde{A}^j x) \tag{3.12}$$

with

$$U_A\chi_k = \exp\left(\frac{2\pi i}{n}(k+\varphi)\right)\chi_k. \tag{3.13}$$

Since the sum over the lengths of all closed orbits of $\tilde{A}$ modulo p is $p^2 = N$, we get a complete set of eigenstates in this way.

We have proven theorem 3.1.

Theorem 3.1. Let $x \in (1/p)\mathbb{Z}^2$ and let n be the smallest positive integer such that (3.10) holds. Then (3.11) is satisfied for some $\varphi \in \mathbb{R}$. The χ_k, $k = 0, 1, \ldots, n-1$, defined by (3.12), are eigenstates of U_A, and the eigenvalues are given by (3.13).

The phase φ can be calculated using the periodicity property

$$\psi(x + py) = \exp(i\pi p x \wedge y)\exp(i\pi y_1 y_2)\psi(x) \qquad x, y \in \frac{1}{p}\mathbb{Z}^2 \qquad py = y_1 e_1 + y_2 e_2. \tag{3.14}$$

4. The Wigner functions of the eigenstates

In this section we will discuss the behaviour of the expectation values

$$\langle\chi|t\left(\frac{1}{p}d\right)|\chi\rangle \qquad d \in \mathbb{Z}^2 \tag{4.1}$$

for eigenstates ψ of U_A in the limit $p \to \infty$.

They can also be written as Fourier components of the Wigner function of χ. First we will prove the following.

Theorem 4.1. Let $A \in \mathrm{SL}(2, \mathbb{Z})$ be hyperbolic, i.e. $|\mathrm{Tr}\, A| > 2$, and satisfying the property (2.18). Let $d \in \mathbb{Z}^2$, $d \neq 0$. Assume that p is a prime number and large enough. Then, for every eigenstate χ of U_A of the form (3.12) holds that

$$|\langle\chi|t\left(\frac{1}{p}\, d\right)|\chi\rangle| \leq \frac{8}{n(p)}. \tag{4.2}$$

Therefore

$$\langle\chi|t\left(\frac{1}{p}\, d\right)|\chi\rangle \to 0 \tag{4.3}$$

for $p \to \infty$, p prime number.

Proof. Since $(\mathrm{Tr}\, \tilde{A})^2 - 4 \neq 0$, p is no divisor of $(\mathrm{Tr}\, \tilde{A})^2 - 4$, if p is large enough.

We will now consider $\tilde{A}$ over the field $\mathbb{Z}_p$, i.e. we replace the matrix elements of $\tilde{A}$ by the corresponding residue classes modulo p. We will denote this new matrix again by $\tilde{A}$.

We have seen that we may assume $(\mathrm{Tr}\, \tilde{A})^2 - 4 \neq 0$ (over $\mathbb{Z}_p$). Therefore $\tilde{A}$ has two different eigenvalues λ and λ^{-1}. They are in $\mathbb{Z}_p$ or in the larger field $\mathbb{Z}_p[\lambda]$. Let

$$\tilde{A} z_1 = \lambda z_1 \qquad \tilde{A} z_2 = \lambda^{-1} z_2. \tag{4.4}$$

Consider now $x \in \mathbb{Z}_p^2$, $x = x_1 z_1 + x_2 z_2$. Then

$$\tilde{A}^n x = x \Leftrightarrow x_1 \lambda^n z_1 + x_2 \lambda^{-n} z_2 = x_1 z_1 + x_2 z_2. \tag{4.5}$$

Therefore $n = n(p)$, if $x \neq 0$ and $p > 2$, and $n(p)$ can be characterised as the smallest positive integer such that

$$\lambda^{n(p)} = 1. \tag{4.6}$$

If χ is of the form (3.12) with $x \in \mathbb{Z}^2$ (and therefore $n = 1$), then $\langle\chi|t((1/p)\, d)|\chi\rangle = 0$, if $d \not\equiv 0 \bmod p$. But this is the case for p large enough. If $x \notin \mathbb{Z}^2$ then we have seen that $n = n(p)$. In order to show (4.2) it is now sufficient to prove that at most eight of the terms $\langle\psi(\tilde{A}^k x)|t((1/p)\, d)|\psi(\tilde{A}^j x)\rangle$, $0 \leq k, j \leq n(p) - 1$, are different from zero. Put $y = px$ and consider y over $\mathbb{Z}_p$. We have then to study the number of solutions k, j of the equation

$$\tilde{A}^k y - \tilde{A}^j y = d. \tag{4.7}$$

Let $y = y_1 z_1 + y_2 z_2$ and $d = d_1 z_1 + d_2 z_2$. Then (4.7) implies

$$(\lambda^k - \lambda^j) y_1 = d_1 \tag{4.8}$$

$$(\lambda^{-k} - \lambda^{-j}) y_2 = d_2. \tag{4.9}$$

We consider first the case $d_1, d_2 \neq 0$. Then we get $d_2 = \lambda^{-k-j}(\lambda^j - \lambda^k) y_2 = -\lambda^{-k-j} d_1 y_2 / y_1$ and therefore

$$\lambda^{k+j} = -d_1 y_2 / d_2 y_1. \tag{4.10}$$

Since $\lambda^n = \lambda^m$ implies $n(p) | m - n$ because of (4.6), and since $0 \leq k + j \leq 2n(p) - 2$, there are at most two possible values for $k + j$ such that (4.10) holds. Multiplication of (4.8) and (4.9) yields

$$(2 - \lambda^{k-j} - \lambda^{j-k}) y_1 y_2 = d_1 d_2. \tag{4.11}$$

This is a quadratic equation in $\mu=\lambda^{k-j}$ having at most two solutions for μ. To each μ there are at most two solutions for $k-j$. Therefore there are at most four possible values for $k-j$ and the number of solutions k, j is not larger than eight.

It remains to consider the case $d_1=0$ or $d_2=0$. Let $d_1=0$ and

$$d=\begin{pmatrix} a \\ b \end{pmatrix} \qquad a, b \in \mathbb{Z}.$$

Then

$$\tilde{A}\begin{pmatrix} a \\ b \end{pmatrix} \equiv \lambda \begin{pmatrix} a \\ b \end{pmatrix} \bmod p. \tag{4.12}$$

It follows that p is a divisor of $(\tilde{A}_{11}-\lambda)a+\tilde{A}_{12}b$ and of $\tilde{A}_{21}a+(\tilde{A}_{22}-\lambda)b$ and therefore of

$$b[(\tilde{A}_{11}-\lambda)a+\tilde{A}_{12}b]-a[\tilde{A}_{21}a+(\tilde{A}_{22}-\lambda)b]=\tilde{A}_{12}b^2+(\tilde{A}_{11}-\tilde{A}_{22})ab-\tilde{A}_{21}a^2. \tag{4.13}$$

Thus (4.13) must be zero if p large enough.

But then

$$[2\tilde{A}_{12}b+(\tilde{A}_{11}-\tilde{A}_{22})a]^2=4\tilde{A}_{12}\tilde{A}_{21}a^2+(\tilde{A}_{11}-\tilde{A}_{22})^2a^2=[(\operatorname{Tr}\tilde{A})^2-4]a^2 \tag{4.14}$$

because of $\det \tilde{A}=1$.

This means that $(\operatorname{Tr}\tilde{A})^2-4$ is a square number, i.e. there exists $m\in\mathbb{N}_0$ such that

$$(\operatorname{Tr}\tilde{A})^2-4=m^2. \tag{4.15}$$

This is a contradiction since $(\operatorname{Tr}\tilde{A})^2=m^2$ or $|(\operatorname{Tr}\tilde{A})^2-m^2|>4$ for $|\operatorname{Tr}\tilde{A}|>2$, and we have proven (4.2). Then (4.3) follows from $n(p)\to\infty$ $(p\to\infty)$. This is so, because $\tilde{A}^{n(p)}\equiv\mathbb{1} \bmod p$ but $\tilde{A}^{n(p)}\neq\mathbb{1}$ and therefore $\|\tilde{A}^{n(p)}\|\geqslant p\to\infty (p\to\infty)$.

Next we will consider what happens if χ is an eigenstate of U_A, but not of the form (3.12). This might be the case if the corresponding eigenvalue is degenerate. Let

$$A^{n(p)}=\mathbb{1}+pB. \tag{4.16}$$

Then it follows from (3.14) that the states (3.12) have the eigenvalues given by (3.13) with

$$\varphi=\tfrac{1}{2}p(x\wedge Bx)+\tfrac{1}{2}a_1a_2 \tag{4.17}$$

where $pBx=a_1e_1+a_2e_2$.

Two states of the form (3.12) associated with x and $\tilde{x}$ can only have the same eigenvalue if $\varphi-\tilde{\varphi}\in\mathbb{Z}$. If we put $y=px$, $w=p\tilde{x}$ this gives

$$\frac{1}{2}\left[\frac{1}{p}(y\wedge By)-\frac{1}{p}(w\wedge Bw)+(a_1a_2-\tilde{a}_1\tilde{a}_2)\right]=k \tag{4.18}$$

for some $k\in\mathbb{Z}$. Therefore

$$p\,|\,(y\wedge By)-(w\wedge Bw). \tag{4.19}$$

Let now χ be an arbitrary eigenstate of U_A given by

$$\chi=\sum\alpha(x)\psi(x). \tag{4.20}$$

Then $\alpha(x)$, $\alpha(\tilde{x})\neq 0$ only if (4.19). Furthermore $|\alpha(x)|\leqslant 1/\sqrt{n(p)}$, if $x\notin\mathbb{Z}^2$.

Fix $x \in (1/p)\mathbb{Z}^2$ and assume $\alpha(x) \neq 0$. Then $\alpha^*(\tilde{x}_2)\alpha(\tilde{x}_1)\langle\psi(\tilde{x}_2)|t((1/p)d)|\psi(\tilde{x}_1)\rangle \neq 0$ only if (4.19) (with $w = p\tilde{x}_1$) and

$$p \mid w \wedge Bd + d \wedge Bw + d \wedge Bd. \tag{4.21}$$

We will now do all calculations over $\mathbb{Z}_p[\lambda]$.

Let $y = y_1 z_1 + y_2 z_2$, $w = w_1 z_1 + w_2 z_2$. Since B commutes with A and $\lambda \neq \lambda^{-1}$, z_1, z_2 are also eigenvectors of B. Let

$$Bz_i = \mu_i z_i \qquad i = 1, 2. \tag{4.22}$$

Then (4.19) and (4.21) are equivalent to

$$y_1\mu_2 y_2 - y_2\mu_1 y_1 = w_1\mu_2 w_2 - w_2\mu_1 w_1 \tag{4.23}$$

$$w_1\mu_2 d_2 - w_2\mu_1 d_1 + d_1\mu_2 w_2 - d_2\mu_1 w_1 + d_1\mu_2 d_2 - d_2\mu_1 d_1 = 0. \tag{4.24}$$

If $\mu_1 \neq \mu_2$ it follows

$$y_1 y_2 = w_1 w_2 \tag{4.25}$$

$$w_1 d_2 + d_1 w_2 + d_1 d_2 = 0. \tag{4.26}$$

Therefore

$$w_1^2 d_2 + d_1 y_1 y_2 + d_1 d_2 w_1 = 0 \tag{4.27}$$

$$w_2^2 d_1 + d_2 y_1 y_2 + d_1 d_2 w_2 = 0. \tag{4.28}$$

These are quadratic equations in w_1 and w_2, respectively (we have seen that we may assume $d_1, d_2 \neq 0$). Thus we obtain at most four solutions for w_1, w_2.

Note that the case $x = \tilde{x}_1 = 0$ cannot occur, since then $y = w = 0$ in contradiction to (4.26). Therefore it holds always $|\alpha^*(\tilde{x}_2)\alpha(\tilde{x}_1)| \leq 1/n(p)$. It remains the case $\mu_1 = \mu_2 =: \mu$.

Then

$$\tilde{A}^{n(p)} \equiv (1 + \mu p) \cdot \mathbb{1} \mod p^2 \tag{4.29}$$

and

$$1 = \det \tilde{A}^{n(p)} \equiv (1 + \mu p)^2 \equiv 1 + 2\mu p \mod p^2. \tag{4.30}$$

This implies $\mu \equiv 0 \mod p$, if $p > 2$. But then

$$\tilde{A}^{n(p)} \equiv \mathbb{1} \mod p^2 \tag{4.31}$$

which is equivalent to $n(p^2) = n(p)$. We cannot treat this special case. The degeneracies are extremely high in this case.

We have proven theorem 4.20.

Theorem 4.2. Under the assumptions of theorem 4.1 but for arbitrary eigenstate χ of U_A holds

$$\left|\langle\chi|t\left(\frac{1}{p}d\right)|\chi\rangle\right| \leq \frac{4}{n(p)} \tag{4.32}$$

if $n(p^2) > n(p)$.

We will now see that our results imply that the corresponding Wigner functions converge weakly to the constant $1/4\pi^2$ in an appropriate topology.

The Wigner function $W(q, p)$ corresponding to the state ψ can be characterised by

$$\int_0^{2\pi}\int_0^{2\pi} \mathrm{d}q\,\mathrm{d}p f(q,p) W(q,p) = \sum_{k,l=-\infty}^{\infty} \hat{f}_{k,l}\langle\psi|\exp[\mathrm{i}(kq+lp)]|\psi\rangle \tag{4.33}$$

where

$$f(q,p) = \sum_{k,l=-\infty}^{\infty} \hat{f}_{k,l}\exp[\mathrm{i}(kq+lp)]. \tag{4.34}$$

We remind the reader that

$$\mathrm{e}^{\mathrm{i}(kq+lp)} = t\left(\frac{1}{\sqrt{N}}(ke_1+le_2)\right). \tag{4.35}$$

It follows immediately from (4.33) that

$$\left|\int_0^{2\pi}\int_0^{2\pi} \mathrm{d}q\,\mathrm{d}p f(q,p) W(q,p)\right| \leqslant \sum_{k,l} |\hat{f}_{k,l}|. \tag{4.36}$$

Therefore we may regard $W(q, p)$ as a linear functional on the Banach space E of all $f:[0, 2\pi]^2 \to \mathbb{C}$ with $\|f\| := \Sigma_{k,l} |\hat{f}_{k,l}| < \infty$, and its norm is bounded by 1. Thus

$$\langle\psi_n|\exp[\mathrm{i}(kq+lp)]|\psi_n\rangle \to \delta_{k,0}\delta_{l,0} \qquad (n\to\infty) \tag{4.37}$$

implies $W_n \to 1/4\pi^2$ in the weak *-topology of E*.

To see the 'δ-brush' structure of the Wigner function as stated in Hannay and Berry (1980) we use the periodicity of $t((1/\sqrt{N})(ke_1+le_2))$ under $k\to k+2N$, $l\to l+2N$. This implies

$$\int_0^{2\pi}\int_0^{2\pi} \mathrm{d}q\,\mathrm{d}p\, f(q,p) W(q,p) = \sum_{k,l=0}^{2N-1} \tilde{f}_{k,l}\langle\psi|\exp[\mathrm{i}(kq+lp)]|\psi\rangle \tag{4.38}$$

where

$$\tilde{f}_{k,l} = \sum_{\mu,\nu=-\infty}^{\infty} \hat{f}_{k+2\mu N, l+2\nu N} = \frac{1}{4N^2}\sum_{m,n=0}^{2N-1} f\left(\frac{\pi m}{N}, \frac{\pi n}{N}\right)\exp\left(-\frac{\mathrm{i}\pi}{N}(km+ln)\right). \tag{4.39}$$

One can identify $W(q, p)$ also as a bounded linear functional on $C([0, 2\pi]^2)$. Let

$$F = \sum_{k,l=0}^{2N-1} \tilde{f}_{k,l}\exp[\mathrm{i}(kq+lp)]. \tag{4.40}$$

Then

$$|\langle\psi|F|\psi\rangle| \leqslant (\langle\psi|F^*F|\psi\rangle)^{1/2} \leqslant (\mathrm{Tr}\, F^*F)^{1/2}. \tag{4.41}$$

Using the fact that $\mathrm{Tr}\, t^*(y)t(x) = 0$ for $x - y \notin \mathbb{Z}^2$ we get

$$\mathrm{Tr}\, F^*F \leqslant \text{constant} \times N \sum_{k,l=0}^{2N-1} |\tilde{f}_{k,l}|^2. \tag{4.42}$$

Since

$$\sum_{k,l=0}^{2N-1} |\tilde{f}_{k,l}|^2 = \frac{1}{4N^2}\sum_{m,n=0}^{2N-1} \left|f\left(\frac{\pi m}{N}, \frac{\pi n}{N}\right)\right|^2 \leqslant \|f\|_\infty^2 \tag{4.43}$$

it follows that

$$|(4.38)| \leqslant \text{constant} \times \sqrt{N}\, \|f\|_\infty. \tag{4.44}$$

Unfortunately this bound goes to infinity for $N \to \infty$, and we cannot show convergence in the weak *-topology corresponding to $C([0, 2\pi]^2)$.

Although we can treat theorems 4.1 and 4.2 only the case $N = (\text{prime})^2$ rigorously, we expect that the Wigner functions converge weakly to a constant also for general N. First we observe that for each eigenstate ψ holds

$$\langle\psi|t\left(\frac{1}{\sqrt{N}} d\right)|\psi\rangle = \langle\psi|G|\psi\rangle \qquad d \in \mathbb{Z}^2 \tag{4.45}$$

where

$$G = \frac{1}{n} \sum_{k=0}^{n-1} t\left(\frac{1}{\sqrt{N}} (A^T)^k d\right) \qquad n = n(N) \tag{4.46}$$

because of (2.8). Now $t((1/\sqrt{N})d)$ and $t((1/\sqrt{N})A^{\mathrm{T}}d)$ commute if and only if $d \wedge A^{\mathrm{T}}d \equiv 0 \bmod N$. But then $d \wedge A^{\mathrm{T}}d = 0$ for N large enough and d has to be an eigenvector of A^{T}. This is impossible, as we have seen in the proof of theorem 4.1. We therefore expect that the operators in the sum (4.46) behave like random unitary matrices. This suggests that

$$\|G\| \sim \frac{1}{\sqrt{n}}. \tag{4.47}$$

Let us also make a remark to the construction of eigenstates in section 3. Here the assumption N = perfect square was crucial. Note, however, that one can make a similar construction for N = prime, for instance. To do this replace the states in (3.3) by the eigenstates of $t((1/\sqrt{N})y)$, $y \in \mathbb{Z}^2$, where y is an eigenvector of A^{T} modulo N (we assume in addition that y exists). Then U_A permutes these states and a complete set of eigenstates can be constructed analogously as in section 3.

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